1 Change of variables in double integrals

Review of the idea of substitution

Consider the integral

$$\int_0^2 x\cos(x^2)dx.$$

To evaluate this integral we use the u-substitution

$$u = x^2.$$

This substitution send the interval $[0,2]$ onto the interval $[0,4]$. Since

$$du = 2xdx \tag{1}$$

the integral becomes

$$\frac{1}{2}\int_0^4 \cos u du = \frac{1}{2}\sin 4.$$

We want to perform similar subsitutions for multiple integrals.

Jacobians

Let

$$x = g(u,v) \quad \text{and} \quad y = h(u,v) \tag{2}$$

be a transformation of the plane. Then the Jacobian of this transformation is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{3}$$

Theorem

Let

$$x = g(u,v) \quad \text{and} \quad y = h(u,v)$$

be a transformation of the plane that is one to one from a region S in the (u,v)-plane to a region R in the (x,y)-plane. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

$$\int\int_R f(x,y)dxdy = \int\int_S f[g(u,v),h(u,v)]|\frac{\partial(x,y)}{\partial(u,v)}|dudv \tag{4}$$

Here $|\dots|$ means the absolute value.

Remark A useful fact is that

$$|\frac{\partial(x,y)}{\partial(u,v)}| = \frac{1}{|\frac{\partial(u,v)}{\partial(x,y)}|} \tag{5}$$

Example
Use an appropriate change of variables to evaluate

$$\int\int_R (x-y)^2 dxdy \tag{6}$$

where R is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$. (excercise: draw the domain R).

Solution:
We find that the equations of the four lines that make the parallelogram are

$$x-y=0 \quad x-y=2 \quad x+y=0 \quad x+y=2 \tag{7}$$

The equations (7) suggest the change of variables

$$u = x-y \quad v = x+y \tag{8}$$

Solving (8) for x and y gives

$$x = \frac{u+v}{2} \quad y = \frac{v-u}{2} \tag{9}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{10}$$

The region S in the (u,v) is the square $0 < u < 2$, $0 < v < 2$. Since $x - y = u$, the integral becomes

$$\int_0^2 \int_0^2 u^2 \frac{1}{2} dudv = \int_0^2 [\frac{u^3}{6}]_0^2 dv = \int_0^2 \frac{4}{3} dv = \frac{8}{3}$$

Polar coordinates
We know describe examples in which double integrals can be evaluated by changing to polar coordinates. Recall that polar coordinates are defined by

$$x = r\cos\theta \quad y = r\sin\theta \tag{11}$$

The Jacobian of the transformation (11) is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \qquad (12)$$

Example
Let R be the disc of radius 2 centered at the origin. Calculate

$$\int\int_R \sin(x^2+y^2)dxdy \qquad (13)$$

Using the polar coordinates (11) we rewrite (13) as

$$\int_0^{2\pi} \sin(r^2)|\frac{\partial(x,y)}{\partial(r,\theta)}|drd\theta \qquad (14)$$

Substituting (12) into (14) we obtain

$$\int_0^{2\pi}\int_0^2 r\sin r^2 drd\theta \qquad (15)$$

Using the subsitutuion $t = r^2$ we have

$$\int_0^{2\pi}\int_0^4 \frac{1}{2}\sin t\, dtd\theta = \pi(1-\cos 4).$$